

ESTIMATES FOR THE FIRST EIGENVALUE OF JACOBI OPERATOR ON HYPERSURFACES

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ABSTRACT. In this paper, we study the first eigenvalue of Jacobi operator on an n -dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$. We give an optimal upper bound for the first eigenvalue of Jacobi operator, which only depends on the mean curvature H and the dimension n . This bound is attained if and only if, $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ when $H \neq 0$ or $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n-1$ when $H = 0$.

1. INTRODUCTION

Let $\varphi : M \rightarrow S^{n+1}(1)$ be an n -dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$ of dimension $n+1$. We consider a variation of the hypersurface $\varphi : M \rightarrow S^{n+1}(1)$, for any $t \in (-\varepsilon, \varepsilon)$,

$$\varphi_t : M \rightarrow S^{n+1}(1)$$

is an immersion with $\varphi_0 = \varphi$. The area of φ_t is given by

$$A(t) = \int_M dA_t$$

and the volume of φ_t is defined by

$$V(t) = \frac{1}{n+1} \int_M \langle \varphi_t, N(t) \rangle dA_t,$$

where $N(t)$ denotes the unit normal of φ_t . For any t , if $V(t) = V(0)$, then the variation φ_t is called volume-preserving. If the variational vector $\frac{\partial \varphi_t}{\partial t}|_{t=0} = fN$ for a smooth function f , then the variation is called a normal variation, where N is the unit normal of φ . Let H denote the mean curvature of φ . The first variation formula of the area functional $A(t)$ is given by

$$\frac{dA(t)}{dt}|_{t=0} = - \int_M nHf dA,$$

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where $f = \langle \frac{\partial \varphi_t}{\partial t} |_{t=0}, N \rangle$. Hence, we know, for a compact minimal hypersurface, that is, $H = 0$

$$\frac{dA(t)}{dt} \Big|_{t=0} = 0,$$

namely, compact minimal hypersurfaces are critical points of the area functional $A(t)$. The second variation formula of $A(t)$ is given by

$$\frac{d^2 A(t)}{dt^2} \Big|_{t=0} = - \int_M f J_m f dA$$

and

$$J_m f = \Delta f + (S + n)f,$$

where S denotes the squared norm of the second fundamental form of φ and Δ stands for the Laplace-Beltrami operator. The J_m is called a Jacobi operator or a stability operator of the minimal hypersurface φ (cf. [8]).

Let $\lambda_1^{J_m}$ denote the first eigenvalue of the Jacobi operator J_m . Then

$$J_m u = -\lambda_1^{J_m} u$$

and the $\lambda_1^{J_m}$ is given by

$$\lambda_1^{J_m} = \inf_{f \neq 0} \frac{- \int_M f J_m f dA}{\int_M f^2 dA}.$$

Simons [9] proves

$$\lambda_1^{J_m} \leq -n$$

and $\lambda_1^{J_m} = -n$ if and only if $\varphi : M \rightarrow S^{n+1}(1)$ is totally geodesic. Furthermore, Wu [10] proves that for an n -dimensional compact non-totally geodesic minimal hypersurface $\varphi : M \rightarrow S^{n+1}(1)$ in $S^{n+1}(1)$, then $\lambda_1^{J_m} \leq -2n$ and $\lambda_1^{J_m} = -2n$ if and only if $\varphi : M \rightarrow S^{n+1}(1)$ is a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n-1$. Thus, we know that the upper bound for the first eigenvalue $\lambda_1^{J_m}$ due to Wu is optimal and it only depends on the dimension n , does not depends on the immersion.

On the other hand, if one considers the volume-preserving variation of φ , then we have

$$\int_M f dA = 0.$$

From the first variation formula:

$$\frac{dA(t)}{dt} \Big|_{t=0} = - \int_M n H f dA,$$

we know that compact hypersurfaces with constant mean curvature are critical points of the area functional $A(t)$ for the volume-preserving variation and the second variation formula of $A(t)$ is given by

$$\frac{d^2 A(t)}{dt^2} \Big|_{t=0} = - \int_M f J_m f dA,$$

where the Jacobi operator J_m of compact hypersurfaces with constant mean curvature is the same as one of compact minimal hypersurfaces ([3]).

Alias, Barros and Brasil [2] study the first eigenvalue of the Jacobi operator J_m of compact hypersurfaces with constant mean curvature. They prove the following:

Theorem ABB. *If $\varphi : M \rightarrow S^{n+1}(1)$ is an n -dimensional compact hypersurface with non-zero constant mean curvature H in the unit sphere $S^{n+1}(1)$, then $\lambda_1^{J_m} = -n(1 + H^2)$ and $\varphi : M \rightarrow S^{n+1}(1)$ is totally umbilical or*

$$\lambda_1^{J_m} \leq -2n(1 + H^2) + \frac{n(n-2)|H|}{\sqrt{n(n-1)}} \max \sqrt{S - nH^2}$$

and the equality holds if and only if $\varphi : M \rightarrow S^{n+1}(1)$ is $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$, with $r^2 > \frac{1}{n}$ for $n \geq 3$.

According to this theorem, we know that, for $n = 2$, the upper bound of the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator of non-totally umbilical compact hypersurfaces with constant mean curvature only depends on the mean curvature H and the dimension. But for $n \geq 3$, the upper bound of the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator on non-totally umbilical compact hypersurfaces with constant mean curvature includes the term $\max \sqrt{S - nH^2}$. Hence, the upper bound of the first eigenvalue $\lambda_1^{J_m}$ does not only depend on the mean curvature H and the dimension n , but also depends on the immersion φ .

It is natural and important to propose the following:

Problem 1.1. To find an optimal upper bound for the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator of non-totally umbilical compact hypersurfaces with constant mean curvature, which only depends on the mean curvature H and the dimension n .

In this paper, we give an affirmative answer for the above problem 1.1.

Theorem 1.1. *Let $\varphi : M \rightarrow S^{n+1}(1)$ be an n -dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$.*

- (1) *If $2 \leq n \leq 4$ or $n \geq 5$ and $n^2 H^2 < \frac{16(n-1)}{n(n-4)}$, then the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies*

$$\lambda_1^{J_m} \leq -n(1 + H^2) - \frac{n(\sqrt{4(n-1) + n^2 H^2} - (n-2)|H|)^2}{4(n-1)}$$

and the equality holds if and only if $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ with $r > 0$ satisfying

$$\begin{cases} 1 > r^2 > \frac{1}{n} & \text{for } 2 \leq n \leq 4, \\ \frac{n}{(n-2)^2} > r^2 > \frac{1}{n}, & \text{for } n \geq 5 \text{ and } n^2 H^2 < \frac{16(n-1)}{n(n-4)} \end{cases}$$

or $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n-1$ with $H = 0$.

- (2) If $n \geq 5$ and $n^2 H^2 \geq \frac{16(n-1)}{n(n-4)}$, the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies

$$\lambda_1^{J_m} \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)} H^2$$

and the equality holds if and only if $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$.

Remark 1.1. Since the first eigenvalue of Jacobi operator J_m on totally umbilical hypersurfaces satisfies $\lambda_1^{J_m} = -n(1+H^2)$, according to our theorem, one knows that for $2 \leq n \leq 4$, there are no n -dimensional compact hypersurfaces in the unit sphere with constant mean curvature H so that the first eigenvalue $\lambda_1^{J_m}$ of Jacobi operator J_m takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}, -n(1+H^2) \right).$$

For any $n \geq 2$, there are no n -dimensional compact hypersurfaces in the unit sphere with constant mean curvature H satisfying $n^2 H^2 < \frac{16(n-1)}{n(n-4)}$ so that the first eigenvalue $\lambda_1^{J_m}$ of Jacobi operator J_m takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}, -n(1+H^2) \right).$$

One should compare the bound

$$-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}$$

with the pinching constant in the rigidity theorem of Cheng and Nakagawa [6] or Alencar and do Carmo [1].

In the section 2, we shall give some examples of compact hypersurfaces, the first eigenvalue of Jacobi operator is given.

2. PRELIMINARY

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let $\varphi : M \rightarrow S^{n+1}(1)$ be an n -dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a local orthonormal frame on M . Hence, we have

$$\omega_{n+1} = 0$$

on M . From Cartan's lemma, we have

$$(2.1) \quad \omega_{in+1} = \sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The mean curvature H and the second fundamental form II of $\varphi : M \rightarrow S^{n+1}(1)$ are defined, respectively, by

$$H = \frac{1}{n} \sum_{i=1}^n h_{ii}, \quad II = \sum_{i,j=1}^n h_{ij} \omega_i \otimes \omega_j \mathbf{e}_{n+1}.$$

When the mean curvature H of $\varphi : M \rightarrow S^{n+1}(1)$ is identically zero, we recall that $\varphi : M \rightarrow S^{n+1}(1)$ is by definition a *minimal hypersurface*. From the structure equations of $\varphi : M \rightarrow S^{n+1}(1)$, Gauss equation is given by

$$(2.2) \quad R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

From (2.2), we have

$$n(n-1)r = n(n-1) + n^2H^2 - S,$$

where $n(n-1)r$ and S denote the scalar curvature and the squared norm of the second fundamental form of $\varphi : M \rightarrow S^{n+1}(1)$, respectively. Defining the covariant derivative of h_{ij} by

$$(2.3) \quad \sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{ik} \omega_{kj} + \sum_k h_{kj} \omega_{ki},$$

we obtain the Codazzi equations

$$(2.4) \quad h_{ijk} = h_{ikj}.$$

By taking exterior differentiation of (2.3), and defining

$$(2.5) \quad \sum_l h_{ijkl} \omega_l = dh_{ijk} + \sum_l h_{ljk} \omega_{li} + \sum_l h_{ilk} \omega_{lj} + \sum_l h_{ijl} \omega_{lk},$$

we have the following Ricci identities:

$$(2.6) \quad h_{ijk} - h_{ikj} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}.$$

For any C^2 -function f on M , we define its gradient and Hessian by

$$df = \sum_{i=1}^n f_i \omega_i,$$

$$\sum_{j=1}^n f_{ij} \omega_j = df_i + \sum_{j=1}^n f_j \omega_{ji}.$$

Thus, the Laplace-Beltrami operator Δ is given by

$$\Delta f = \sum_{i=1}^n f_{ii}.$$

Example 2.1. For totally umbilical sphere $S^n(r)$ of radius $r > 0$, the first eigenvalue $\lambda_1^{J_m} = -n(1 + H^2)$ with $H = \frac{1}{r}$.

Example 2.2. For Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, $k = 1, 2, \dots, n$, the first eigenvalue $\lambda_1^{J_m} = -2n$ with $H = 0$.

Example 2.3. For hypersurfaces $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ with $0 < r < 1$, the principal curvatures are given by

$$k_1 = -\frac{\sqrt{1-r^2}}{r}, \quad k_2 = \dots = k_n = \frac{r}{\sqrt{1-r^2}}.$$

Hence, we know that

$$nH = \frac{nr^2 - 1}{r\sqrt{1-r^2}}, \quad S = \frac{1 - 2r^2 + nr^4}{r^2(1-r^2)}.$$

For $r^2 \geq \frac{1}{n}$, by a direct computation, we know that the first eigenvalue $\lambda_1^{J_m}$ of $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$\lambda_1^{J_m} = -n(1 + H^2) - \frac{n(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|)^2}{4(n-1)}.$$

For $n \geq 5$ and $\frac{1}{n} \leq r^2 < \frac{n}{(n-2)^2}$, we know the hypersurface $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$n^2H^2 < \frac{16(n-1)}{n(n-4)}$$

and

$$\lambda_1^{J_m} = -n(1 + H^2) - \frac{n(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|)^2}{4(n-1)}.$$

The hypersurface $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$ satisfies

$$\lambda_1^{J_m} = -2(n-1)(1 + H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

with $n^2H^2 = \frac{16(n-1)}{n(n-4)}$.

3. PROOF OF THEOREM 1.1.

In this section, we give a proof of the theorem 1.1.

Proof of theorem 1.1. By making use of the Codazzi equations, Ricci identities and a standard computation of Simons' type formula (cf. [6], [4, 5], [7] and [9]), we have

$$(3.1) \quad \frac{1}{2}\Delta S = \sum_{i,j,k=1}^n h_{ijk}^2 + nS - n^2H^2 + nHf_3 - S^2,$$

where $f_3 = \sum_{i=1}^n k_i^3$ and k_i , $i = 1, 2, \dots, n$ denote the principal curvatures. Putting $\mu_i = k_i - H$, we have

$$(3.2) \quad B := \sum_{i=1}^n \mu_i^2 = S - nH^2 \geq 0, \quad f_3 = B_3 + 3HB + nH^3,$$

where $B_3 = \sum_{i=1}^n \mu_i^3$. The following inequality is known (cf. [6] and [7]):

$$(3.3) \quad |B_3| \leq \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}},$$

and the equality holds if and only if at least $n-1$ of k_i , for $i = 1, 2, \dots, n$, are equal with each other. Since H is constant, we can assume $H \geq 0$. Thus, from (3.1), (3.2) and (3.3), we have

$$(3.4) \quad \frac{1}{2} \Delta B = \frac{1}{2} \Delta S \geq \sum_{i,j,k=1}^n h_{ijk}^2 + B(n + nH^2 - B) - nH \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}}.$$

For any constant $\alpha > 0$ and $\varepsilon > 0$, we consider a function $f_\varepsilon = (B + \varepsilon)^\alpha > 0$. Hence, we have, from (3.4),

$$(3.5) \quad \begin{aligned} \Delta f_\varepsilon &= \alpha(\alpha-1)(B+\varepsilon)^{\alpha-2} |\nabla B|^2 + \alpha(B+\varepsilon)^{\alpha-1} \Delta B \\ &\geq \alpha(\alpha-1)(B+\varepsilon)^{\alpha-2} |\nabla B|^2 \\ &\quad + 2\alpha(B+\varepsilon)^{\alpha-1} \left(\sum_{i,j,k=1}^n h_{ijk}^2 + B(n + nH^2 - B) - nH \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}} \right). \end{aligned}$$

Since H is constant, we have

$$(3.6) \quad \begin{aligned} \nabla_k(nH) &= \sum_{i=1}^n h_{iik} = 0, \quad h_{kkk}^2 \leq (n-1) \sum_{i \neq k} h_{iik}^2 \\ |\nabla B|^2 &= \sum_{k=1}^n \left(2 \sum_{i=1}^n \mu_i h_{iik} \right)^2 \leq 4B \sum_{i,k=1}^n h_{iik}^2. \end{aligned}$$

Thus, we obtain

$$(3.7) \quad \begin{aligned} |\nabla B|^2 &\leq 4B \sum_{i,k=1}^n h_{iik}^2 \\ &= 4B \left(\frac{n}{n+2} \sum_{k=1}^n h_{kkk}^2 + \frac{2}{n+2} \sum_{k=1}^n h_{kkk}^2 + \sum_{i \neq k} h_{iik}^2 \right) \\ &\leq \frac{4n}{n+2} B \left(\sum_{k=1}^n h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 \right). \end{aligned}$$

For any constant β , we have

$$\lambda_1^{J_m} \int_M f_\varepsilon^2 dA \leq - \int_M f_\varepsilon J_m f_\varepsilon dA$$

$$\begin{aligned}
&= -\beta \int_M f_\varepsilon \Delta f_\varepsilon dA - \int_M \left((1-\beta) f_\varepsilon \Delta f_\varepsilon + (S+n) f_\varepsilon^2 \right) dA \\
&= \beta \int_M |\nabla f_\varepsilon|^2 dA - \int_M f_\varepsilon \left\{ (1-\beta) \left(\alpha(\alpha-1)(B+\varepsilon)^{\alpha-2} |\nabla B|^2 \right. \right. \\
&\quad \left. \left. + \alpha(B+\varepsilon)^{\alpha-1} \Delta B \right) + (B+nH^2+n) f_\varepsilon \right\} dA \\
&= \alpha \int_M f_\varepsilon \{1+2\alpha\beta-\beta-\alpha\} (B+\varepsilon)^{\alpha-2} |\nabla B|^2 dA \\
&\quad - \int_M f_\varepsilon^2 \left\{ \frac{\alpha(1-\beta)}{B+\varepsilon} \Delta B + B+nH^2+n \right\} dA.
\end{aligned}$$

Since

$$\sum_{i,j,k=1}^n h_{ijk}^2 = \sum_{k=1}^n h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk}^2,$$

from (3.7), for α and β , which satisfy

$$\alpha > \frac{n-2}{4n}, \quad 1-\beta = \frac{2n\alpha}{4n\alpha+2-n},$$

we obtain

$$\begin{aligned}
(3.8) \quad & (1+2\alpha\beta-\beta-\alpha) |\nabla B|^2 - 2(1-\beta)(B+\varepsilon) \sum_{i,j,k=1}^n h_{ijk}^2 \\
& \leq \frac{2}{n+2} B \left\{ (n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha \right\} \left(\sum_{k=1}^n h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 \right) = 0.
\end{aligned}$$

Thus, we infer

$$\begin{aligned}
& \lambda_1^{J_m} \int_M f_\varepsilon^2 dA \\
& \leq \alpha \int_M f_\varepsilon (B+\varepsilon)^{\alpha-2} \left\{ (1+2\alpha\beta-\beta-\alpha) |\nabla B|^2 - 2(1-\beta)(B+\varepsilon) \sum_{i,j,k=1}^n h_{ijk}^2 \right\} dA \\
& \quad - \int_M f_\varepsilon^2 \left\{ \frac{2\alpha(1-\beta)B}{B+\varepsilon} \left((n+nH^2-B) - nH \frac{(n-2)}{\sqrt{n(n-1)}} B^{\frac{1}{2}} \right) + B+nH^2+n \right\} dA \\
& \leq - \int_M f_\varepsilon^2 \left\{ \frac{B}{B+\varepsilon} \left(\{1-2\alpha(1-\beta)\} B - \frac{2\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}} nHB^{\frac{1}{2}} + \varepsilon \right) \right. \\
& \quad \left. - 2\alpha(1-\beta)(n+nH^2) \right\} \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} dA - (n+nH^2) \int_M f_\varepsilon^2 dA.
\end{aligned}$$

For $1 - 2\alpha(1 - \beta) > 0$, we obtain

$$\begin{aligned} & \lambda_1^{J_m} \int_M f_\varepsilon^2 dA \\ & \leq \int_M f_\varepsilon^2 \left\{ \frac{B}{B + \varepsilon} \left(\frac{\alpha^2(1 - \beta)^2(n - 2)^2}{(1 - 2\alpha(1 - \beta))n(n - 1)} (nH)^2 - \varepsilon \right) \right. \\ & \quad \left. - 2\alpha(1 - \beta)(n + nH^2) \int_M f_\varepsilon^2 \frac{B}{B + \varepsilon} dA - (n + nH^2) \int_M f_\varepsilon^2 dA \right\} dA. \end{aligned}$$

Since $\varphi : M \rightarrow S^{n+1}(1)$ is not totally umbilical, we have

$$\lim_{\varepsilon \rightarrow 0} \int_M f_\varepsilon^2 dA = \int_M B^{2\alpha} dA > 0.$$

Letting $\varepsilon \rightarrow 0$, we derive

$$(3.9) \quad \lambda_1^{J_m} \leq -(1 + 2\alpha(1 - \beta))n(1 + H^2) + \frac{\alpha^2(1 - \beta)^2}{1 - 2\alpha(1 - \beta)} \frac{(n - 2)^2}{n(n - 1)} n^2 H^2.$$

If $2 \leq n \leq 4$ or $n \geq 5$ and $n^2 H^2 < \frac{16(n-1)}{n(n-4)}$, we have

$$\frac{1}{2} \left(1 - \sqrt{\frac{(n - 2)^2 H^2}{4(n - 1) + n^2 H^2}} \right) \geq \frac{1}{2} - \frac{1}{n}.$$

By taking

$$\alpha(1 - \beta) \rightarrow \frac{1}{2} \left(1 - \sqrt{\frac{(n - 2)^2 H^2}{4(n - 1) + n^2 H^2}} \right),$$

we obtain

$$\lambda_1^{J_m} \leq -n(1 + H^2) - \frac{n}{4(n - 1)} (\sqrt{4(n - 1) + n^2 H^2} - (n - 2)|H|)^2.$$

If the equality holds and $H = 0$, we have $\lambda_1^{J_m} = -2n$. Hence $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n - 1$. If $H \neq 0$ and the equality holds, we know that $h_{ijk} = 0$, for any $i, j, k = 1, 2, \dots, n$. Hence, we know that the second fundamental form is parallel and S is constant. Thus, we know that $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1 - r^2})$ since, from the (3.3), the $n - 1$ of the principal curvatures are equal with each other. From the examples in the section 2, we know that r satisfies

$$\begin{cases} r^2 > \frac{1}{n} & \text{for } 2 \leq n \leq 4, \\ \frac{1}{n} < r^2 < \frac{n}{(n - 2)^2}, & \text{for } n \geq 5 \text{ and } n^2 H^2 < \frac{16(n - 1)}{n(n - 4)}. \end{cases}$$

If $n \geq 5$ and $n^2 H^2 \geq \frac{16(n-1)}{n(n-4)}$, we take

$$\alpha(1 - \beta) = \frac{1}{2} - \frac{1}{n}.$$

Thus,

$$\lambda_1^{J_m} \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

If the equality holds, we know

$$(1-2\alpha(1-\beta))\sqrt{B} = \frac{\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}}nH.$$

Thus, we have

$$(3.10) \quad S = B + nH^2 = nH^2 + \frac{(n-2)^4}{16n(n-1)}n^2H^2.$$

because of

$$\alpha(1-\beta) = \frac{1}{2} - \frac{1}{n}.$$

Since S is constant, the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator is given by

$$\lambda_1^{J_m} = -S - n = -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

Hence, we obtain

$$(3.11) \quad S = n-2 + 2(n-1)H^2 - \frac{(n-2)^4}{8(n-1)}H^2.$$

From (3.10) and (3.11), we get

$$n-2 = (2-n)H^2 + \frac{(n-2)^4(n+2)}{16(n-1)}H^2,$$

$$1 = \frac{n(n-4)}{16(n-1)}n^2H^2,$$

that is,

$$n^2H^2 = \frac{16(n-1)}{n(n-4)}.$$

Since, from the (3.3), the $n-1$ of the principal curvatures are equal with each other, From the examples in the section 2, we know that $\varphi : M \rightarrow S^{n+1}(1)$ is isometric to $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$. It completes the proof of theorem 1.1.

□

REFERENCES

1. Alencar, H. and do Carmo, M. hypersurfaces with constant mean curvature, Proc. Amer. Math. Soc. , **120**(1994), 11223-1229.
2. Alías, L. J., Barros, A. & Brasil, A. Jr, A spectral characterization of $H(r)$ -torus by the first stability eigenvalue, Proc. Amer. Math. Soc., **133**(2005), 875-884.
3. Barbosa, J. L., do Carmo, M. & Eschenburg, J., Stability of hypersurfaces with constant mean curvature in Riemannian Manifolds, Math. Z., **197**(1988), 123-138.

4. Cheng, Q. -M., The rigidity of Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$, *Comment. Math. Helv.*, **71**(1996), 60-69.
5. Cheng, Q. -M., Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature, *J. London Math. Soc.*, **64**(2001), 755-768.
6. Cheng, Q. -M. & Nakagawa, H., Totally umbilical hypersurfaces, *Hiroshima Math. J.*, **20**(1990), 1-10.
7. Li, H., Hypersurfaces with constant scalar curvature in space forms, *Math. Ann.*, **305**(1996), 665-672.
8. Perdomo, O., First stability eigenvalue characterization of Clifford hypersurfaces, *Proc. Amer. Math. Soc.*, **130**(2002), 3379-3384
9. Simons, J., Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88**(1968), 62-105.
10. Wu, C., New characterizations of the Clifford tori and the Veronese surface, *Arch. Math. (Basel)*, **61**(1993), 277-284

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